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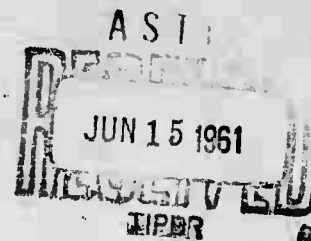
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# A RAND REPORT

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## ADVANCED RESEARCH PROJECTS AGENCY

DEPARTMENT OF DEFENSE

ON THE DIFFUSION MATRIX OF RADIATIVE TRANSFER

Sueo Ueno\*

RM-2744-ARPA

March 9, 1961

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SUMMARY

The author has previously used the stochastic model of radiative transfer to obtain the diffusion matrix consisting of the reflectance and the transmittance operators in a finite plane-parallel atmosphere of arbitrary stratification. When this method is applied to the distribution of emission sources within the medium, the emergent intensities can be computed without actually solving the equation of transfer.

In the present paper, using auxiliary equations in conjunction with the Milne integral equations, the author derives the diffusion matrix along with the extension concerning the Neumann solution as given by Busbridge. In the case of diffuse reflection and transmission of parallel rays, the solutions are expressed in terms of a pair of scattering and transmission functions for each of the two boundaries of the atmosphere. Then these global functions are given by  $X^*$  and  $Y^*$  functions that are equal to those previously found by Bellman and Kalaba. Whereas the diffusion matrix formally has a somewhat similar appearance to a map yielded by Preisendorfer, the mathematical development is different.

If the optical properties of the medium are constant throughout the atmosphere, the reflectance and transmittance operators obtained here reduce to those given by Sobolev.

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## ON THE DIFFUSION MATRIX OF RADIATIVE TRANSFER

I. INTRODUCTION

In 1942 Ambarzumian [1] published a mathematical method for solving the homogeneous first integral equation of Milne. Basing his work on the idea of linear aggregation, he reduced the solution of the Milne equation to that of its auxiliary equation, which in turn leads directly to the  $H$  equation (see Kourganoff [22]). In a second paper, Ambarzumian [2] treated the same problem from a physical viewpoint. By means of the principle of invariance, he obtained a nonlinear integral equation in the  $H$  function directly through physical analysis of the diffuse reflection and transmission of light, without actually solving the homogeneous Milne equation for the source function. Ingeniously extending the invariance method, Chandrasekhar [15] formulated a complete set of principles of invariance in a finite homogeneous atmosphere and applied it to various transfer problems. The requisite integral equations for the scattering and transmission functions were derived from the above set of invariance principles in connection with the equation of transfer.

By means of the Ambarzumian technique, the solution of the nonhomogeneous Milne equation can be expressed in terms of the  $H$  function for certain special forms of the distribution of emission sources acting in the medium. The technique plays an important role in the solutions of transfer problems from the practical viewpoint. Furthermore, as Busbridge [13] emphasized,

the technique becomes more powerful when it is connected with the theory of the N solution. The auxiliary equation for any given problem reduces to the first integral equation of Milne for the diffuse reflection and transmission of a parallel beam of radiation by a finite layer. From the mathematical point of view, the Ambarzumian technique was ingeniously extended by Busbridge [12,13] and was applied to various transfer problems of astrophysical interest by Busbridge [12,13] and Ueno [29].

Ambarzumian's physical method based on the principle of invariance has been applied by Busbridge [11,14] and Stibbs [14,25] to problems of line formation in the Milne-Eddington model with coherent and noncoherent scattering.

Recently, developing the idea of the invariance principle of Ambarzumian [2], Bellman and Kalaba [3] stated the principle of invariant imbedding. The functional relationships among the members of the class are found by imbedding the original process within a family of processes of similar nature and obtaining an invariant process. Whereas the classical approaches reduce problems to the solutions of systems of linear equations, the invariant-imbedding technique reduces problems to the iteration of nonlinear transformations. The principle of invariant imbedding led not only to new analytical functions of radiative transfer (Bellman and Kalaba [2,3]), neutron diffusion (Bellman, Kalaba, and Wing [8,10]), random walk and scattering (Bellman and Kalaba [6]), adaptive processes and random transmission (Bellman and Kalaba [7]), and

problems of Stefan type (Bellman and Kalaba [9]), but also to those of wave propagation (Bellman and Kalaba [5]).

By means of the Chandrasekhar-Wick method, and with the aid of Chandrasekhar's extension of the physical method based on the invariance principle, the following kinds of transfer problems have been solved: line formation in planetary atmosphere (Chandrasekhar [15]), line formation in the Schuster model (Chandrasekhar [15]), emitting atmosphere (Horak [18], Horak and Lundquist [19]), and molecular absorbing atmosphere (King [21]).

Allowing for the map consisting of complete reflectance and complete transmittance operators in inhomogeneous one-parameter carrier space, Preisendorfer [23] obtained the invariant imbedding relation for radiative transfer and neutron transport contexts. Furthermore, the functional relations for reflectance and transmittance operators were derived in a manner similar to that used by Chandrasekhar for the homogeneous case [24]. The polarity of the two operators is elucidated because of the inhomogeneous optical properties of the medium.

In recent years, introducing the probability concept into the theory of radiative theory, Sobolev [27] has treated many subjects from the statistical point of view, including various kinds of transfer problems: pure scattering [27], line formation with coherent and noncoherent scattering in a semi-infinite homogeneous medium [27], diffuse reflection in a semi-infinite inhomogeneous medium [26], diffuse reflection in a finite homogeneous medium [28], and others [27].

Recently, assuming that multiple scattering of a photon as the carrier of radiant energy is a random process of Markovian type, Ueno used a stochastic approach in the study of various problems of radiative transfer: Milne's problem [30], line formation in semi-infinite atmospheres with coherent and noncoherent scattering [31,32], diffuse reflection and transmission in a finite homogeneous and inhomogeneous layer [33,34], and a Markovian property of radiative transfer [35]. Through the probabilistic technique, the integro-differential equation for the emission probability distribution can be derived from the Chapman-Kolmogoroff equation, and the integral transform of the probabilistic equation can be used to obtain the angular distributions of the emergent radiations without actually solving the equation of transfer. It is of interest to mention that the derivation of the S and T functions from the probabilistic equation by the stochastic approach is similar in part to that used recently by Feller [16] on boundaries and lateral conditions for the Kolmogoroff differential equation.

In a preceding paper [37], based on a stochastic model of radiative transfer, the author obtained the diffusion matrix consisting of reflectance and transmittance operators. In the present paper, however, we derive the diffusion matrix from the auxiliary equations that play a major role in the Ambarzumian technique in connection with the Milne equations (cf. Hopf [17], Busbridge [13]). In subsequent papers, the diffusion matrix

will be applied to the various transfer problems of current interest in a finite inhomogeneous medium.

Whereas, in the context of the operational form, the map due to Preisendorfer [23] seems to be somewhat similar to the diffusion matrix, the mathematical procedure is different. In contrast to Preisendorfer's approach, which can be patterned after the procedure used by Chandrasekhar for a finite homogeneous atmosphere, our technique is based on the use of auxiliary equations for the source functions.

Finally, it should be mentioned that, in addition to the methods stated above, the Laplace-transform method (see Hopf [17], Kourganoff [22], Huang [22], and Busbridge [13]) is also used in transfer problems.

## II. THE EQUATION OF TRANSFER

Let  $I_\nu(\tau_\nu, \mu)$  be the specific intensity of  $\nu$  radiation at the optical depth  $\tau_\nu$  in the direction  $\cos^{-1}\mu$ . In considering the transfer problem of radiation in a plane-parallel inhomogeneous atmosphere of finite optical thickness  $\tau_{\nu,1}$ , for simplicity we shall restrict our discussion to coherent and isotropic scattering. Then, for convenience, we suppressed the subscript  $\nu$  of the various quantities.

The equation of transfer appropriate to the present case is written in the form

$$(2.1) \quad \mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) - J(\tau),$$

where the source function  $J(\tau)$  is

$$(2.2) \quad J(\tau) = \frac{1}{2} w(\tau) \int_{-1}^{+1} I(\tau, \mu') d\mu' + B_1(\tau).$$

In equation (2.2),  $w(\tau)$  is the albedo for single scattering, and  $B_1(\tau)$  is the distribution of emission sources within the atmosphere.

Following the notation of Chandrasekhar in [15], we shall denote the intensity of radiation at the level  $\tau$  directed toward the surface  $\tau = 0$  by  $I(\tau, +\mu)$ ,  $0 < \mu \leq 1$ , and that directed toward the surface  $\tau = \tau_1$  by  $I(\tau, -\mu)$ ,  $0 < \mu \leq 1$ . Let radiation of intensity  $I(0, -\mu)$ ,  $0 < \mu \leq 1$ , be incident on the surface  $\tau = 0$  in the direction  $-\mu$ , and let radiation of intensity  $I(\tau_1, +\mu)$ ,  $0 < \mu \leq 1$ , be incident similarly on the surface  $\tau = \tau_1$  in the direction  $+\mu$ .

Then, allowing for the formal solution of equation (2.1) subject to the boundary conditions given above, namely,

$$(2.3) \quad I(\tau, +\mu) = I(\tau_1, +\mu) \exp\left(-\frac{\tau_1 - \tau}{\mu}\right) + \int_{\tau}^{\tau_1} J(t) \exp\left(-\frac{t - \tau}{\mu}\right) \frac{dt}{\mu},$$

$$(2.4) \quad I(\tau, -\mu) = I(0, -\mu) \exp\left(-\frac{\tau}{\mu}\right) + \int_0^{\tau} J(t) \exp\left(-\frac{\tau - t}{\mu}\right) \frac{dt}{\mu},$$

from (2.1) we get the first integral equation of Milne governing  $J(\tau)$ :

$$(2.5) \quad [1 - w(\tau)\Lambda]_{\tau} \{J(t)\} = B(\tau),$$

where  $1$  is the identity operator,  $\Lambda$  is the truncated Hopf operator

$$(2.6) \quad \Lambda_{\tau} \{f(t)\} = \frac{1}{2} \int_0^{\tau_1} f(t) E_1(|t - \tau|) dt,$$

and  $B(\tau)$  is

$$(2.7) \quad B(\tau) = B_1(\tau) + \frac{1}{2} w(\tau) \int_0^1 I(0, -\mu') \exp(-\frac{\tau}{\mu'}) d\mu' \\ + \frac{1}{2} w(\tau) \int_0^1 I(\tau_1, +\mu') \exp(-\frac{\tau_1 - \tau}{\mu'}) d\mu'.$$

In equation (2.5),  $E_1(\tau)$  is the first exponential integral:

$$(2.8) \quad E_1(\tau) = \int_0^1 \exp(-\frac{\tau}{\mu}) \frac{d\mu}{\mu}.$$

While in the homogeneous case  $B(\tau)$  is a known function in any particular problem, the quantity  $B_1(\tau)$  in the case of inhomogeneous distribution of emission sources is given by

$$(2.9) \quad B_1(\tau) = \int_0^1 \frac{d\mu}{\mu} \left[ \int_{\tau}^{\tau_1} B_1(t, +\mu) \exp(-\frac{t - \tau}{\mu}) dt \right. \\ \left. + \int_0^{\tau} B_1(t, -\mu) \exp(-\frac{\tau - t}{\mu}) dt \right].$$

### III. THE AUXILIARY EQUATIONS

In the theory of radiative transfer based on the Ambarzumian technique, the auxiliary equation plays an important role.



In the case of a finite inhomogeneous atmosphere, the auxiliary equations are expressed in the forms

$$(3.1) \quad [1 - w(\tau)\bar{\Lambda}]_{\tau} \{p(\mu; \tau, \tau_1)\} = w(\tau) \exp(-\frac{\tau}{\mu}),$$

$$(3.2) \quad [1 - w(\tau)\bar{\Lambda}]_{\tau} \{p^*(\mu; \tau, \tau_1)\} = w(\tau) \exp(-\frac{\tau_1 - \tau}{\mu}).$$

Physically, the function  $p(\mu; \tau, \tau_1)$  corresponds to the source function due to a parallel beam of radiation incident on the surface  $\tau = 0$  in the direction  $-\mu$ , and  $p^*(\mu; \tau, \tau_1)$  corresponds to that due to similar parallel rays on the surface  $\tau = \tau_1$  in the direction  $+\mu$ . From the probabilistic viewpoint, the function  $p(\mu; \tau, \tau_1)$  represents the probability that a photon absorbed at the level  $\tau$  will reappear in the direction  $+\mu$  as radiation emerging from the surface  $\tau = 0$ , and  $p^*(\mu; \tau, \tau_1)$  is the probability that a photon absorbed at the level  $\tau$  will be reemitted in the direction  $-\mu$  as radiation escaping from the surface  $\tau = \tau_1$ . Mathematically, the probability  $p(\mu; \tau, \tau_1)d\mu$  is the probability of finding  $\mu$  in the range  $(\mu, \mu + d\mu)$  at the level  $\tau$ .

On differentiating the auxiliary equations with respect to  $\tau_1$ , after some argument we have

$$(3.3) \quad \frac{\partial p(\mu; \tau, \tau_1)}{\partial \tau_1} = \frac{1}{2} p(\mu; \tau_1, \tau_1) \int_0^1 p^*(\mu'; \tau, \tau_1) \frac{d\mu'}{\mu'},$$

$$(3.4) \quad \frac{\partial p^*(\mu; \tau, \tau_1)}{\partial \tau_1} = -\frac{1}{\mu} p^*(\mu; \tau, \tau_1) + \frac{1}{2} p^*(\mu; \tau_1, \tau_1) \int_0^1 p^*(\mu'; \tau, \tau_1) \frac{d\mu'}{\mu'}.$$

Equations (3.3) and (3.4) can also be obtained by means of the probabilistic method [34] and the invariant imbedding technique [36]. Furthermore, the solutions are found to depend on four functions  $X(\mu, \tau_1)$ ,  $Y(\mu, \tau_1)$ ,  $X^*(\mu, \tau_1)$ , and  $Y^*(\mu, \tau_1)$ ; which are connected to each other by the principle of reciprocity (see [34]). Mathematically speaking, the source functions  $p(\mu; \tau, \tau_1)$  and  $p^*(\mu; \tau, \tau_1)$  are the  $N$  solutions of the auxiliary equations (3.1) and (3.2) (see Busbridge [13]).

#### IV. DIFFUSION MATRIX

With the aid of equations (2.3) and (2.4), the emergent intensities from the atmosphere  $I(0, +\mu)$  and  $I(\tau_1, -\mu)$  are respectively provided by

$$(4.1) \quad I(0, +\mu) = I(\tau_1, +\mu) \exp(-\frac{\tau_1}{\mu}) + I^*(0, +\mu),$$

$$(4.2) \quad I(\tau_1, -\mu) = I(0, -\mu) \exp(-\frac{\tau_1}{\mu}) + I^*(\tau_1, -\mu),$$

where

$$(4.3) \quad I^*(0, +\mu) = \int_0^{\tau_1} J(t) \exp(-\frac{t}{\mu}) \frac{dt}{\mu},$$

$$(4.4) \quad \begin{aligned} I^*(\tau_1, -\mu) &= \int_0^{\tau_1} J(t) \exp(-\frac{\tau_1 - t}{\mu}) \frac{dt}{\mu} \\ &= \int_0^{\tau_1} J(\tau_1 - t) \exp(-\frac{t}{\mu}) \frac{dt}{\mu}. \end{aligned}$$

Following Busbridge [13], we call equations (4.3) and (4.4) the  $\tau_1$  transforms of  $J(\tau)$ .

When

$$(4.5) \quad B(\tau) = aB^1(\tau) + bB^2(\tau_1 - \tau),$$

and

$$(4.6) \quad [1 - w(\tau)\bar{\Lambda}]_{\tau}[J^1(t)] = B^1(\tau),$$

$$(4.7) \quad [1 - w(\tau)\bar{\Lambda}]_{\tau}[J^2(t)] = B^2(\tau_1 - \tau),$$

then the solution of equation (2.5) is written in the form

$$(4.8) \quad J(\tau) = aJ^1(\tau) + bJ^2(\tau).$$

Furthermore, the emergent intensities  $I^*(0, +\mu)$  and  $I^*(\tau_1, -\mu)$  are given by

$$(4.9) \quad I^*(0, +\mu) = a \int_0^{\tau_1} J^1(t) \exp\left(-\frac{t}{\mu}\right) \frac{dt}{\mu} \\ + b \int_0^{\tau_1} J^2(t) \exp\left(-\frac{t}{\mu}\right) \frac{dt}{\mu},$$

$$(4.10) \quad I^*(\tau_1, -\mu) = a \int_0^{\tau_1} J^1(t) \exp\left(-\frac{\tau_1 - t}{\mu}\right) \frac{dt}{\mu} \\ + b \int_0^{\tau_1} J^2(t) \exp\left(-\frac{\tau_1 - t}{\mu}\right) \frac{dt}{\mu}.$$

On multiplying equation (4.6) by  $p(\mu; \tau, \tau_1)/w(\tau)\mu$  and integrating with respect to  $\tau$  over  $(0, \tau_1)$ , we get

$$(4.11) \quad \int_0^{\tau_1} [p(\mu; \tau, \tau_1) \frac{J^1(\tau)}{w(\tau)} - p(\mu; \tau, \tau_1) \bar{\Lambda}_{\tau}[J^1(t)]] \frac{d\tau}{\mu} \\ = \int_0^{\tau_1} p(\mu; \tau, \tau_1) \frac{B^1(\tau)}{w(\tau)} \frac{d\tau}{\mu}.$$

On the other hand, if we multiply equation (3.1) by  $J^1(\tau)/w(\tau)\mu$  and integrate with respect to  $\tau$  over  $(0, \tau_1)$ , we get

$$(4.12) \quad \int_0^{\tau_1} \left[ \frac{J^1(\tau)}{w(\tau)} p(\mu; \tau, \tau_1) - J^1(\tau) \bar{\Lambda}_\tau \{ p(\mu; \tau, \tau_1) \} \right] \frac{d\tau}{\mu} \\ = \int_0^{\tau_1} J^1(\tau) \exp\left(-\frac{\tau}{\mu}\right) \frac{d\tau}{\mu}.$$

Allowing for the symmetrical property of the  $\bar{\Lambda}$  operator (cf. Busbridge [13]), we have

$$\int_0^{\tau_1} f_1(t) \bar{\Lambda}_t \{ f_2(t') \} dt = \int_0^{\tau_1} f_2(t) \bar{\Lambda}_t \{ f_1(t') \} dt,$$

and using equations (4.11) and (4.12), we obtain

$$(4.13) \quad \int_0^{\tau_1} J^1(\tau) \exp\left(-\frac{\tau}{\mu}\right) \frac{d\tau}{\mu} = \int_0^{\tau_1} p(\mu; \tau, \tau_1) \frac{B^1(\tau)}{w(\tau)} \frac{d\tau}{\mu}.$$

Similarly, using equations (3.1), (3.2), (4.6), and (4.7), we get

$$(4.14) \quad \int_0^{\tau_1} J^1(\tau) \exp\left(-\frac{\tau_1 - \tau}{\mu}\right) \frac{d\tau}{\mu} = \int_0^{\tau_1} p^*(\mu; \tau, \tau_1) \frac{B^1(\tau)}{w(\tau)} \frac{d\tau}{\mu},$$

$$(4.15) \quad \int_0^{\tau_1} J^2(\tau) \exp\left(-\frac{\tau}{\mu}\right) \frac{d\tau}{\mu} = \int_0^{\tau_1} p(\mu; \tau_1 - \tau, \tau_1) \frac{B^2(\tau)}{w(\tau_1 - \tau)} \frac{d\tau}{\mu},$$

$$(4.16) \quad \int_0^{\tau_1} J^2(\tau_1 - \tau) \exp\left(-\frac{\tau}{\mu}\right) \frac{d\tau}{\mu} = \int_0^{\tau_1} p^*(\mu; \tau_1 - \tau, \tau_1) \frac{B^2(\tau)}{w(\tau_1 - \tau)} \frac{d\tau}{\mu}.$$

Then, using equations (4.9), (4.10), and (4.13)-(4.16), we can write the emergent intensities  $I^*(0, +\mu)$  and  $I^*(\tau_1, -\mu)$  in the forms

$$(4.17) \quad I^*(0, +\mu) = a \int_0^{\tau_1} p(\mu; \tau, \tau_1) \frac{B^1(\tau)}{w(\tau)} \frac{d\tau}{\mu} \\ + b \int_0^{\tau_1} p(\mu; \tau_1 - \tau, \tau_1) \frac{B^2(\tau)}{w(\tau_1 - \tau)} \frac{d\tau}{\mu},$$

$$(4.18) \quad I^*(\tau_1, -\mu) = a \int_0^{\tau_1} p^*(\mu; \tau, \tau_1) \frac{B^1(\tau)}{w(\tau)} \frac{d\tau}{\mu} \\ + b \int_0^{\tau_1} p^*(\mu; \tau_1 - \tau, \tau_1) \frac{B^2(\tau)}{w(\tau_1 - \tau)} \frac{d\tau}{\mu}.$$

From equations (4.17) and (4.18) the emergent intensities are given in terms of the diffusion matrix as follows:

$$(4.19) \quad \begin{pmatrix} I(0, +\mu) \\ I(\tau_1, -\mu) \end{pmatrix} = D_\mu \begin{pmatrix} L_0(\tau) \\ L_1(\tau) \end{pmatrix},$$

where

$$(4.20) \quad L_0(\tau) = \frac{aB^1(\tau)}{w(\tau)}, \quad L_1(\tau) = \frac{bB^2(\tau)}{w(\tau_1 - \tau)}.$$

In equation (4.19), the diffusion matrix  $D_\mu$  is written in the form

$$(4.21) \quad D_\mu = \begin{pmatrix} \mathcal{R}_\mu & \mathcal{I}_\mu^* \\ \mathcal{I}_\mu & \mathcal{R}_\mu^* \end{pmatrix},$$

where the reflectance and the transmittance operators are

$$(4.22) \quad \mathcal{R}_\mu[f(t)] = \int_0^{\tau_1} p(\mu; t, \tau_1) f(t) \frac{dt}{\mu},$$

$$(4.23) \quad \mathcal{I}_\mu[f(t)] = \int_0^{\tau_1} p^*(\mu; t, \tau_1) f(t) \frac{dt}{\mu},$$

$$(4.24) \quad \mathcal{K}_{\mu}^*[f(t)] = \int_0^{\tau_1} p(\mu; \tau_1 - t, \tau_1) f(t) \frac{dt}{\mu},$$

$$(4.25) \quad \mathcal{L}_{\mu}^*[f(t)] = \int_0^{\tau_1} p(\mu; \tau_1 - t, \tau_1) f(t) \frac{dt}{\mu}.$$

Thus the integral operational matrix  $\mathcal{O}_{\mu}$  given by the probabilistic method in the preceding paper [37] is now derived from the auxiliary equations and the first integral equation of Milne.

As a special case, we shall consider the diffuse reflection and transmission of a parallel beam of radiation by a finite inhomogeneous atmosphere.

Let a parallel beam of radiation of intrinsic flux  $F_0$  fall on the surface  $\tau = 0$  in the direction  $-\mu_0$ , and let a parallel beam of radiation of intrinsic flux  $F_1$  fall on the surface  $\tau = \tau_1$  in the direction  $+\mu_1$ . Then, recalling equation (4.20),

$$L_0(\tau) = \frac{F_0}{4} \exp(-\frac{\tau}{\mu_0}), \quad L_1(\tau) = \frac{F_1}{4} \exp(-\frac{\tau}{\mu_1}),$$

and using equation (4.19), we determine that the emergent intensities are given by

$$(4.26) \quad \begin{aligned} L(\mu, +\mu) = & \frac{1}{2} F_1 \delta(\mu - \mu_1) \exp(-\frac{\tau_1}{\mu_1}) \\ & + \frac{F_0}{4} \int_0^{\tau_1} p(\mu; \tau, \tau_1) \exp(-\frac{\tau}{\mu_0}) \frac{d\tau}{\mu} \\ & + \frac{F_1}{4} \int_0^{\tau_1} p(\mu; \tau_1 - \tau, \tau_1) \exp(-\frac{\tau}{\mu_1}) \frac{d\tau}{\mu}, \end{aligned}$$

$$\begin{aligned}
 (4.27) \quad I(\tau_1, -\mu) &= \frac{1}{2} F_0 \delta(\mu - \mu_0) \exp(-\frac{\tau_1}{\mu_0}) \\
 &+ \frac{F_0}{4} \int_0^{\tau_1} p^*(\mu; \tau, \tau_1) \exp(-\frac{\tau}{\mu_0}) \frac{d\tau}{\mu} \\
 &+ \frac{F_1}{4} \int_0^{\tau_1} p^*(\mu; \tau_1 - \tau, \tau_1) \exp(-\frac{\tau}{\mu_1}) \frac{d\tau}{\mu}.
 \end{aligned}$$

Writing

$$(4.28) \quad \int_0^{\tau_1} p(\mu; \tau, \tau_1) \exp(-\frac{\tau}{\mu_0}) d\tau = S(\tau_1; \mu_0, \mu),$$

$$(4.29) \quad \int_0^{\tau_1} p(\mu; \tau_1 - \tau, \tau_1) \exp(-\frac{\tau}{\mu_1}) d\tau = T(\tau_1; \mu_1, \mu),$$

$$(4.30) \quad \int_0^{\tau_1} p^*(\mu; \tau_1 - \tau, \tau_1) \exp(-\frac{\tau}{\mu_1}) d\tau = S^*(\tau_1; \mu_1, \mu),$$

$$(4.31) \quad \int_0^{\tau_1} p^*(\mu; \tau, \tau_1) \exp(-\frac{\tau}{\mu_0}) d\tau = T^*(\tau_1; \mu_0, \mu),$$

and allowing for the principle of reciprocity (cf. [34]), i.e.,

$$(4.32) \quad \begin{cases} S(\tau_1; \mu, \mu_0) = S(\tau_1; \mu_0, \mu), \\ S^*(\tau_1; \mu; \mu_1) = S^*(\tau_1; \mu_1, \mu), \\ T(\tau_1; \mu, \mu_0) = T^*(\tau_1; \mu_0, \mu), \end{cases}$$

we obtain

$$\begin{aligned}
 (4.33) \quad I(0, +\mu) &= \frac{1}{2} F_1 \delta(\mu - \mu_1) \exp(-\frac{\tau_1}{\mu_1}) \\
 &+ \frac{F_0}{4\mu} S(\tau_1; \mu, \mu_0) + \frac{F_1}{4\mu} T^*(\tau_1; \mu, \mu_1),
 \end{aligned}$$

$$(4.34) \quad \dot{i}(\tau_1, -\mu) = \frac{1}{2} F_0 \delta(\mu - \mu_0) \exp\left(-\frac{\tau_1}{\mu_0}\right) + \frac{F_0}{4\mu} T(\tau_1; \mu, \mu_0) + \frac{F_1}{4\mu} S^*(\tau_1; \mu, \mu_1).$$

The emergent intensities (4.33) and (4.34) are equal to those given in the preceding paper [37]. Furthermore, if the optical properties of the medium are constant throughout the atmosphere, the source functions  $p(\mu; \tau, \tau_1)$  and  $p^*(\mu; \tau, \tau_1)$  are equal respectively to  $p^*(\mu; \tau_1 - \tau, \tau_1)$  and  $p(\mu; \tau_1 - \tau, \tau_1)$ . In this case, the reflectance and transmittance operators (4.22) and (4.23) reduce to those yielded by Sobolev [27].



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